

ZADANIE 15 (7 PKT)

Rozpatrujemy wszystkie trapezy $ABCD$, których wierzchołki A i B leżą na wykresie funkcji $f(x) = -\frac{2}{3} \cdot \frac{1}{x^2} - 1$ określonej dla $x \neq 0$. Punkt C ma współrzędne $(1, 1)$, a oś Oy jest osią symetrii tego trapezu (zobacz rysunek).

Oblicz obwód tego trapezu $ABCD$, którego pole jest najmniejsze możliwe.

$D(x, 0)$

$C(1, 1)$ $D(-1, 1)$ $B(x, -\frac{2}{3} \cdot \frac{1}{x^2} - 1)$
 $A(-x, -\frac{2}{3} \cdot \frac{1}{x^2} - 1)$

$|\overline{DC}| = 2$ $|\overline{AB}| = 2x$ $h = |y_C - y_D| =$
 $|\frac{2}{3} \cdot \frac{1}{x^2} + 2| = \frac{2}{3} \cdot \frac{1}{x^2} + 2$

$\begin{cases} P(x) = (1+x) \cdot (\frac{2}{3} \cdot \frac{1}{x^2} + 2) = 2x + 2 + \frac{2}{3} \cdot \frac{1}{x} + \frac{2}{3} \cdot \frac{1}{x^2} \\ x \in \mathbb{R}_+ \end{cases}$

$P'(x) = 2 - \frac{2}{3} \cdot \frac{1}{x^2} - \frac{4}{3} \cdot \frac{1}{x^3} = \frac{6x^3 - 2x - 4}{3x^3}$ $(x^{-2})' = -2x^{-3} = -\frac{2}{x^3}$

$P'(x) = \frac{2(x-1)(3x^2+3x+2)}{3x^3}$

$x \in (0, 1) \quad | \quad 1 \quad | \quad (1+\infty)$
 $+$ $-$ 0 $+$
 x
 $+$ $-$ $+$

$y_{\min}$ dla $x = 1$

$B(1, -\frac{5}{3})$
 $A(-1, -\frac{5}{3})$

$Obw = 9\frac{1}{3}$

6.67. $y = \frac{x^2 - 2x + 3}{x^2 + 2x - 3}$

6.68. $y = \frac{3}{(1-x^2)(1-2x^3)}$

6.69. $y = \frac{\sqrt[3]{x}}{1 - \sqrt[3]{x}} = \frac{x^{\frac{1}{3}}}{1 - x^{\frac{1}{3}}}$

6.70. $z = \frac{1 + \sqrt{t}}{1 + \sqrt{2t}}$

6.71. $s = (3t + 1)^7$

6.72. $v = (4z^2 - 5z + 13)^5$

6.73. $x = \left(\frac{1}{t} + 4\right)^4$

6.74. $s = \left(7t^2 - \frac{4}{t} + 6\right)^6$

$y' = \frac{\frac{1}{x^{\frac{2}{3}}} - \frac{1}{x^{\frac{1}{3}}}}{3 \cdot (1 - x^{\frac{1}{3}})^2} = \frac{x^{\frac{1}{3}} - x^{\frac{2}{3}}}{3(x^{\frac{1}{3}} - x^{\frac{2}{3}})^2}$

$y = \frac{(x^{\frac{1}{3}})' \cdot (1 - x^{\frac{1}{3}}) - (1 - x^{\frac{1}{3}})' \cdot x^{\frac{1}{3}}}{(1 - x^{\frac{1}{3}})^2} =$

$\frac{\frac{1}{3}x^{-\frac{2}{3}}(1 - x^{\frac{1}{3}}) + \frac{1}{3}x^{-\frac{1}{3}} \cdot x^{\frac{1}{3}}}{(1 - x^{\frac{1}{3}})^2} = \frac{x^{-\frac{2}{3}} \cdot x^{\frac{1}{3}} + \frac{1}{3}}{3(1 - x^{\frac{1}{3}})^2 \cdot x^{\frac{1}{3}}}$

$\frac{1}{3(x^{\frac{1}{3}} - x^{\frac{2}{3}})^2}$

$$s = (3t+1)^7. \quad s' = 7(3t+1)^6 \cdot (3t+1)' = \underline{7(3t+1)^6 \cdot 3}$$

6.90. $s = \sqrt{\frac{1-\sqrt{t}}{1+\sqrt{t}}}$. $s'(t) = \frac{1}{2\sqrt{\frac{1-\sqrt{t}}{1+\sqrt{t}}}} \cdot \left(\frac{1-\sqrt{t}}{1+\sqrt{t}}\right)' = \mathbb{D}: t \geq 0, t < 1$

$$\frac{\frac{\sqrt{1-\sqrt{t}}}{2\sqrt{1-\sqrt{t}}} \cdot (1-\sqrt{t})'(1+\sqrt{t}) - (1+\sqrt{t})'(1-\sqrt{t})}{2(\sqrt{1-t} \cdot (1+\sqrt{t}))} = \frac{\frac{1}{2\sqrt{1-t}}(1+\sqrt{t}) - \frac{1}{2\sqrt{t}}(1-\sqrt{t})}{2(\sqrt{1-t} \cdot (1+\sqrt{t}))} = -\frac{1}{2\sqrt{t}(\sqrt{1-t}(1+\sqrt{t}))}$$

6.105. $y = \cos x - \frac{1}{3} \cos^3 x$.

6.107. $y = \operatorname{tg}^4 \sqrt{x}$.

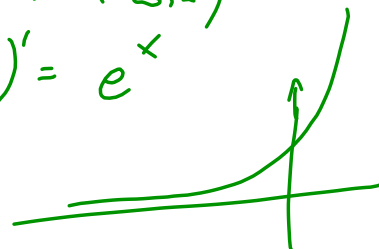
6.109. $y = e^{ax} (a \sin x - \cos x)$.

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(\operatorname{tg} x)' = \left(\frac{\sin x}{\cos x} \right)' = \dots$$

$$(e^x)' = e^x$$



6.109. $y' = (e^{ax} (a \sin x - \cos x))' =$

$$(e^{ax})' (a \sin x - \cos x) + e^{ax} (a \sin x - \cos x)' =$$

$$e^{ax} \cdot a \cdot (a \sin x - \cos x) + e^{ax} (a \cos x + \sin x) =$$

$$= e^{ax} a^2 \sin x - \cancel{ae^{ax} \cos x} + \cancel{e^{ax} \cos x} + e^{ax} \sin x =$$

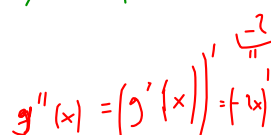
$$= e^{ax} \sin x (a^2 + 1)$$

$$\int e^{ax} \sin(a^2 + 1) dx = e^{ax} (a \sin x - \cos x) + C$$

Diagram showing the relationship between the derivative and the integral:

$$\begin{array}{c} \xrightarrow{+} \\ \int f(x) dx \end{array}$$





ZADANIE 10.19. Zbadać przebieg zmienności funkcji $y = \sin^2 x + \cos x$.

1) $\mathbb{D}: x \in \mathbb{R}$

2) $\lim_{x \rightarrow \pm\infty} (\sin^2 x + \cos x)$ - nie istnieje $\Rightarrow$ $0 = \frac{0}{0}$

3) asymptoty $\rightarrow$ pionowa $x = a$
 $y = ax + b$ $\lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} = \lim_{x \rightarrow \pm\infty} \frac{\sin^2 x}{x} + \lim_{x \rightarrow \pm\infty} \frac{\cos x}{x} = 0$

$a = 0$

$b = \lim_{x \rightarrow \pm\infty} (f(x) - ax) = \lim_{x \rightarrow \pm\infty} f(x)$ - nie istnieje.

1) OY: $x \rightarrow 0 \Rightarrow f(x) = 1$ OY: $(0, 1)$

OX: $\sin^2 x + \cos x = 0$ $1 - \cos^2 x + \cos x = 0$

5) P NP $f(x) = \sin^2 x + \cos x$

7P $f(-x) = \sin^2(-x) + \cos(-x) = \sin^2 x + \cos x = f(x)$

 f' f''

x	
x'	
x''	

10.93. $y = \frac{(x+1)^2}{2x}$ ✓

10.95. $y = \frac{x^2 + 2x + 25}{(x+1)^2}$

10.97. $y = \frac{15x^2 - 13x - 20}{8x^2 + 10x - 7}$

10.99. $y = \frac{x^3}{(x-1)^2}$

10.101. $y = x + 2\sqrt{-x}$ ✓

10.103. $y = 1 - \sqrt[3]{(x-4)^2}$

10.94. $y = \frac{x^2 - x - 4}{x - 1}$

10.96. $y = \frac{x^2 + x + 1}{x^2 - 1}$

10.98. $y = \frac{x^4}{2 - x^3}$

10.100. $y = \frac{(x+3)^3}{(x+2)^2}$

10.102. $y = \sqrt[3]{x^2} - 1$

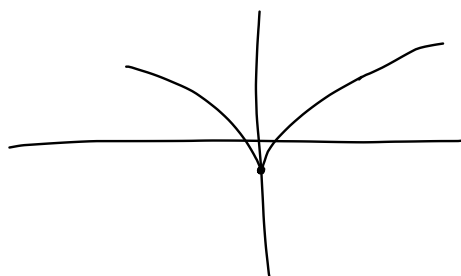
10.104. $y = 2x - 3\sqrt[3]{x^2}$

10.102. $y = \sqrt[3]{x^2} - 1$. $D_f: x \in \mathbb{R}$.

$f'(x) = \frac{2}{3\sqrt[3]{x}}$ $D_{f'}: x \in \mathbb{R} \setminus \{0\}$

$f''(x) = -\frac{2}{3\sqrt[3]{x^4}} < 0$ $D_{f''}$ $\sqrt[3]{-8} = -2$

x	$(-\infty, 0)$	0	$(0, +\infty)$
f'	$-$	$\times$	$+$
f''	$-$	$\times$	$-$
$f(x)$	$\cup$	-1	$\cap$

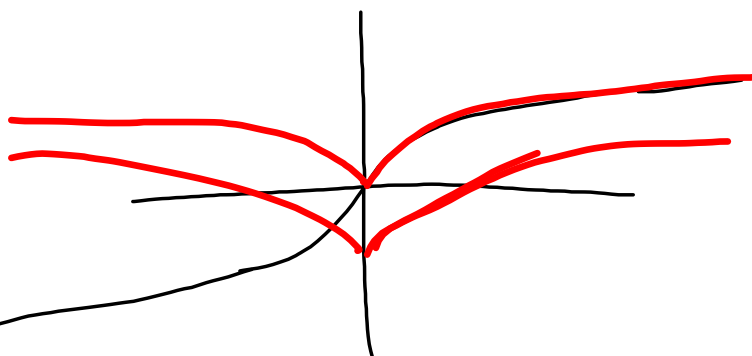


10.102. $y = \sqrt[3]{x^2} - 1$.

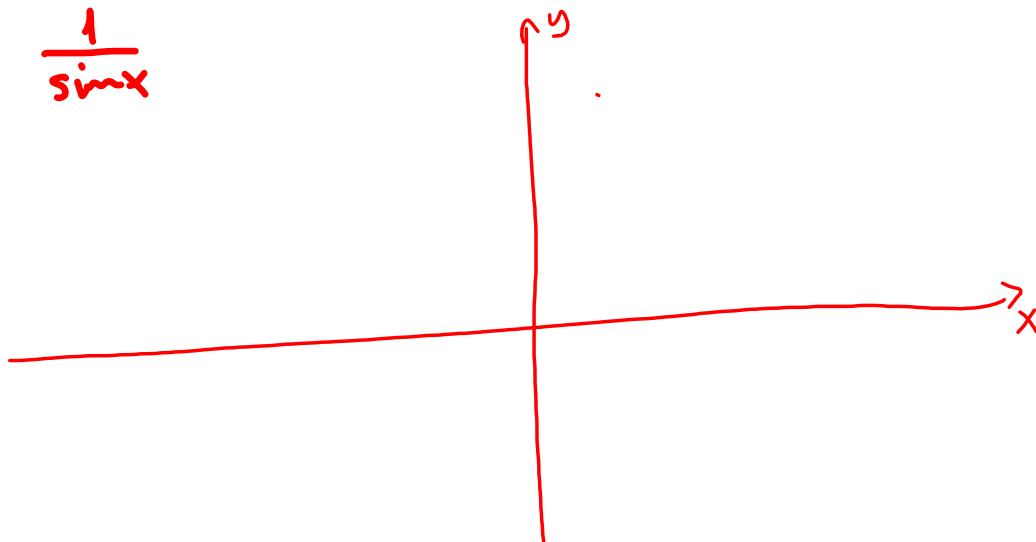
$y = \sqrt[3]{x}$

$\downarrow$

$y = \sqrt[3]{x^2} \approx |\sqrt[3]{x}|$



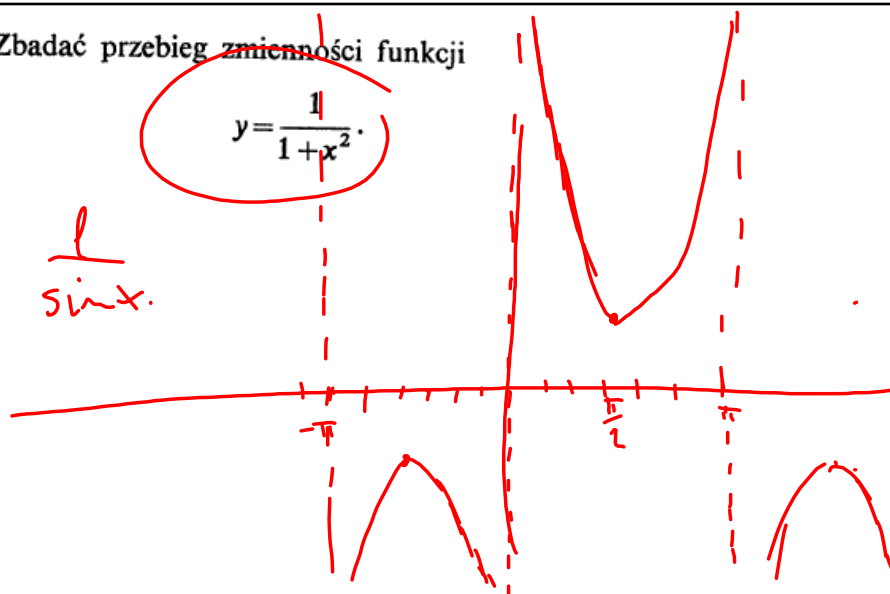
$$y = \frac{1}{\sin x}$$



ZADANIE 10.12. Zbadać przebieg zmienności funkcji

$$y = \frac{1}{1+x^2}$$

$$z = \frac{1}{\sin x}$$



[illegible]

$$\lim_{x \rightarrow x_0} f(x) = \pm \infty \Rightarrow \text{as } x \rightarrow x_0$$

$$\lim_{x \rightarrow +\infty} \sqrt{x} = b \Rightarrow \text{as pos't } \boxed{y = b}$$

$$\left. \begin{array}{l} \lim_{x \rightarrow \pm \infty} \frac{f(x)}{x} = a \in \mathbb{R} \\ \lim_{x \rightarrow \pm \infty} (f(x) - ax) = b \end{array} \right\} \Rightarrow y = ax + b \text{ as. w. l. s. m.}$$

